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It is shown that non-negativity of the solution of the heat equation, given non-negative initial values, and suitable boundary conditions, can be established quite readily once the existence of a solution of the equation depending continuously upon the initial values has been demonstrated.

It is shown that this property is trivially true for the solution of the appropriate finite difference approximation to the partial differential equation, and that the convergence of the solution of the finite difference equation to the solution of the original equation is quite simple, under the above conditions.

ON THE NON-NEGATIVITY OF SOLUTIONS OF THE HEAT EQUATION

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§1. INTRODUCTION

In treating a functional equation, once the question of existence and uniqueness has been disposed of, we turn to a more precise study of the analytic character of the solution. It frequently happens that a method which works very efficiently to establish existence and uniqueness does not yield other properties of the solution in any ready fashion. Conversely, methods which yield non-negativity, convexity, and so forth, may not be ideally suited for the establishment of the basic properties. However, a combination of several techniques may yield the results we desire quite easily.

To illustrate these remarks, let us consider the heat equation

$$u_t = u_{xx} + q(x,t)u,$$

$$(1) \qquad u(x,0) = v(x), \quad 0 \le x \le 1,$$

$$u(0,t) = u(1,t) = 0, \quad t > 0.$$

We shall assume that we have demonstrated, by some means or other, the existence of a solution which depends continuously upon $v(x)$ in the L^2-norm for $t \geq 0$, and that this solution is unique. As we shall see, a method based upon

finite differences will enable us to demonstrate the fact that this solution is non-negative for $t \geq 0$, provided that it is non-negative at $t = 0$, i.e. provided that $v(x) \geq 0$, for $0 \leq x \leq 1$. On the other hand, the particular proof used to establish existence and uniqueness may not have yielded non-negativity in a simple fashion, and, as is known, an existence and uniqueness proof based upon finite differences is not a completely simple matter.

§2. $u_t = u_{xx}$

To illustrate our ideas, begin with the simpler equation

$$(1) \qquad u_t = u_{xx}$$

and consider the difference scheme

$$(2) \qquad \begin{aligned} &(a) \quad w(x, t + \delta^2/2) = [w(x+\delta, t) + w(x-\delta, t)]/2, \\ &(b) \quad w(x, 0) = v(x), \end{aligned}$$

where x takes the values $0, \delta, 2\delta, \ldots, 1$, and t assumes the values $0, \delta^2/2, \delta^2, \ldots$. The function $w(x,t)$ is defined by linearity at non-lattice points.

It is easy to see that, formally, the recurrence relation approaches the partial differential equation as $\delta \to 0$.

As mentioned above, a rigorous proof that the solution of (2) converges to the solution of (1) as $\delta \to 0$, starting from first principles, is non-trivial. However, as we shall see below, a proof of this fact is quite simple, once we have

established the existence and uniqueness of a solution. The fact that $w(x,t)$ is non-negative for any $\delta > 0$ is immediate, and this yields the conclusion that $u(x,t) \geq 0$.

Since we have assumed the existence of a solution of (1) which is a continuous function of $v(x)$, there is no loss of generality in assuming, for our current purposes, that $v(x)$ possesses appropriate continuity properties, sufficient to ensure that

$$(3) \qquad \operatorname*{Max}_{R} \left[|u_{tt}(x,t)|, \quad |u_{xxxx}(x,t)| \right] \leq m < \infty ,$$

where R is the bounded region $0 \leq x \leq 1$, $0 \leq t \leq T < \infty$. We may for example take $v(x)$ to be a trigonometric polynomial. Under the assumption of (2), it is easy to show that in R

$$(4) \qquad \lim_{\delta \to 0} w(x,t) = u(x,t).$$

We have, by virtue of (2)

$$(5) \qquad u(x,t + \delta^2/2) = [u(x+\delta,t) + u(x-\delta,t)]/2 + \delta^4 r(x,t),$$

where $|r(x,t)| \leq 2m$ in R. Consequently, the function $z(x,t) = w(x,t) - u(x,t)$ satisfies the recurrence relation of (1a) with the initial condition $z(x,t) = 0$. Let

$$(6) \qquad d(t) = \operatorname*{Max}_{0 \leq x \leq 1} |w(x,t) - u(x,t)|.$$

Using the recurrence relation, we see that

(7) $$d(t+\delta^2/2) \leq d(t) + 2\delta^4 m$$

for $t = 0, \delta^2/2, \delta^2, \ldots,$ whence

(8) $$d(t) \leq 2\delta^4 mN$$

for $0 \leq t \leq \delta^2 N$. Let $\delta^2 N = T$. Then

(9) $$d(t) \leq 2\delta^2 mT$$

for x and t in R.

From this, we obtain the desired result as $\delta \to 0$.

Note that we can also conclude from the foregoing result that $u(x,t)$ is concave in x for any value of t if $v(x)$ is concave in x. This, in turn, implies that $u(x,t)$ is decreasing in t for each fixed value of x.

§3. $u_t = u_{xx} + q(x,t)u$

To extend the same argument to the general equation of (1.1), we employ the recurrence relation

(1) $$w(x,t+\delta^2/2) = \frac{w(x+\delta,t) + w(x-\delta,t)}{2} + \int_{x-q(x,t)\delta^2/2}^{x+q(x,t)\delta^2/2} u(y,t)\,dy.$$

If we assume that $q(x,t) \geq 0$ for $0 \leq x \leq 1$, $t \geq 0$, the recurrence relation above shows that $w(x,t) \geq 0$ for all x and t. The proof that $w(x,t)$ converges to $u(x,t)$ as $\delta \to 0$ follows the same lines as before.

To see that it is sufficient to assume that

$$(2) \qquad q(x,t) \geq -\lambda > -\infty, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T,$$

for any $T > 0$, where $\lambda = \lambda(T)$, we proceed as follows. Write

$$(3) \qquad u = e^{-\lambda t} v.$$

Then the equation of (1.1) becomes

$$(4) \qquad v_t = v_{xx} + (q(x,t)+\lambda)v,$$

with the same boundary conditions. The new function

$$(5) \qquad q_1(x,t) = q(x,t) + \lambda$$

is non-negative.

§4. GENERALIZATIONS

It is clear that the same method may be employed to obtain corresponding non-negativity results for the solution of the heat equation for higher dimensions and arbitrary regions. The essential part of the proof is the à priori demonstration of the existence and uniqueness of a solution depending continuously upon the initial values, in an appropriate metric.

Similarly, a number of corresponding results can be established for various classes of nonlinear equations.

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